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Shellability and freeness of continuous splines

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ABSTRACT

We provide an example of a shellable polyhedral complex $\mathcal{P}$ in $\mathbb{R}^2$ such that the module of splines $C^0(\hat{\mathcal{P}})$ is not a free module over the polynomial ring in three variables, answering a question raised by Schenck (in press) in [10].

1. Introduction

The study of the space $C^r(\Delta)$ of piecewise polynomial functions (splines) continuously differentiable of order r on a pure d-dimensional simplicial complex $\Delta \subset \mathbb{R}^d$ is a fundamental topic in numerical analysis and approximation theory (see [5]). A main question is to determine the dimension of the vector space $C^r_k(\Delta)$ of splines of degree at most k. One can study all the $C^r_k(\Delta)$ at once by a coning construction: let $\mathbb{R}^d$ have coordinates $x_1, \ldots, x_d$, $\mathbb{R}^{d+1}$ have coordinates $x_0, x_1, \ldots, x_d$, and embed Δ in the hyperplane $x_0 = 1$ of $\mathbb{R}^{d+1}$ by sending $(x_1, \ldots, x_d)$ to $(1, x_1, \ldots, x_d)$. Define $\hat{\Delta} \subset \mathbb{R}^{d+1}$ to be the cone over the image of Δ in $\mathbb{R}^{d+1}$. In [2], Billera and Rose show that $C^r(\hat{\Delta})$ is a graded algebra over the polynomial ring $S = \mathbb{R}[x_0, \ldots, x_d]$ with Hilbert series $HS(C^r(\hat{\Delta}), t) = \sum_{k\geq 0} \dim_{\mathbb{R}} C^r_k(\Delta)t^k = \frac{P(t)}{(1-t)^{d+1}}$, where $P(t) \in \mathbb{Z}[t]$. Much research has focused on characterizing freeness of $C^r(\hat{\Delta})$ and the coefficients of the Hilbert series $HS(C^r(\hat{\Delta}), t)$ in terms of the combinatorics and geometry of Δ [2–4,11,12].

In the case $r = 0$, Billera resolves these questions for simplicial Δ in [4] by showing that $C^0(\hat{\Delta}) \cong A_\Delta$, where A_Δ is the Stanley–Reisner ring of Δ. The Hilbert polynomial of $C^0(\hat{\Delta})$ is determined by the f-vector of Δ ([1], Corollary 3.17) and $C^0(\hat{\Delta})$ is free if and only if A_Δ is Cohen–Macaulay (see Section 2).

If Δ is replaced by a pure d-dimensional polytopal complex $\mathcal{P} \subset \mathbb{R}^d$, $C^r(\hat{\mathcal{P}})$ is more subtle and less studied. In [2], Billera and Rose determine the first two coefficients of the Hilbert polynomial of $C^r(\hat{\mathcal{P}})$ and in [7], MacDonald and Schenck determine the third coefficient. In [13], Yuzvinsky uses cohomology of sheaves on posets to characterize the projective dimension of $C^r(\hat{\mathcal{P}})$. Interest in $C^0(\hat{\mathcal{P}})$ was increased by Payne's result in [9], showing that $C^0(\hat{\mathcal{P}})$ is the equivariant Chow ring of the toric variety associated to the fan $\hat{\mathcal{P}}$. In [10], Schenck analyzes the homology of a certain chain complex and uses a spectral sequence to analyze freeness of $C^0(\hat{\mathcal{P}})$.

In this paper, we focus on freeness of $C^0(\hat{\mathcal{P}})$ as an S-module. If Δ is a simplicial complex, *shellability* of Δ implies that $C^0(\hat{\Delta})$ is free (see Section 2). In [10] Schenck asks the following question.

Question 1.1. *If $\mathcal{P} \subset \mathbb{R}^d$ is a shellable polytopal complex, is $C^0(\hat{\mathcal{P}})$ free?*

Our main result is Example 3.1 which yields the following theorem.

Theorem 1.2. *For a pure, shellable, d-dimensional polytopal complex $\mathcal{P} \subset \mathbb{R}^d$ with $d \geq 2$, freeness of $C^0(\hat{\mathcal{P}})$ as an S-module depends on the embedding of $\mathcal{P}$ in $\mathbb{R}^d$.*

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2. Preliminaries

Let $\mathcal{P}$ be a finite, pure, two dimensional hereditary polytopal complex (in the sense of [14], Section 8.1) supported on $|\mathcal{P}| \subset \mathbb{R}^2$. Let $\mathcal{P}_i$ denote the set of i-dimensional faces of $\mathcal{P}$. A maximal polytope of $\mathcal{P}$ under inclusion is a *facet*; $\mathcal{P}$ is *pure* if all facets have the same dimension. A pure two dimensional complex $\mathcal{P} \subset \mathbb{R}^2$ is *hereditary* iff $\mathcal{P}$ is a decomposition of a 2-manifold with boundary ([4] Lemma 3.3); in higher dimensions the hereditary condition is not so strong. For $U \subset \mathbb{R}^3$ let $C(U)$ denote continuous functions on U. Finally, let $S = \mathbb{R}[x, y, z]$ and let S_k be the vector space of polynomials in S of degree exactly k.

Definition 2.1. Continuous splines on $\hat{\mathcal{P}}$

$$C^0(\hat{\mathcal{P}}) = \{F \in C(|\hat{\mathcal{P}}|) \text{ such that } F|_{\hat{\sigma}} \in S, \forall \sigma \in \mathcal{P}_2\}$$
$$C^0(\hat{\mathcal{P}})_k = \{F \in C(|\hat{\mathcal{P}}|) \text{ such that } F|_{\hat{\sigma}} \in S_k, \forall \sigma \in \mathcal{P}_2\}.$$

In fact $C^0(\hat{\mathcal{P}}) = \bigoplus_{k\in\mathbb{Z}} C^0(\hat{\mathcal{P}})_k$ and from this it is clear that $C^0(\hat{\mathcal{P}})$ is a graded module. $C^0(\hat{\mathcal{P}})$ can be computed as the kernel of a map between free S-modules as follows. Let τ be an edge of $\mathcal{P}$ and $l_\tau \in S$ be the linear form whose vanishing locus is the linear span of $\hat{\tau} \subset \mathbb{R}^3$. Let f denote the number of facets of $\mathcal{P}$, e^0 the number of interior edges, and v^0 the number of internal vertices. The following is Proposition 4.3 of [2].

Proposition 2.2. *$C^0(\hat{\mathcal{P}})$ fits into the graded exact sequence:*

$$0 \to C^0(\hat{\mathcal{P}}) \to S^f \oplus S(-1)^{e^0} \xrightarrow{\phi} S^{e^0} \to M \to 0$$
$$\text{where } \phi = \left(\begin{array}{c|ccc} & l_{\tau_1} & & \\ \delta_d & & \ddots & \\ & & & l_{\tau_{e^0}} \end{array} \right).$$

Here $M = \operatorname{coker} \phi$ and the matrix δ_d is the top dimensional cellular boundary map of $\mathcal{P}$ in relative homology.

The computations in Section 3 are based on this proposition (see 8.3 of [6] for more examples). Note $C^0(\hat{\mathcal{P}})$ is given as a submodule of S^{f+e^0}. It is easily checked that projection onto S^f is injective on the submodule $C^0(\hat{\mathcal{P}})$, so we can view $C^0(\hat{\mathcal{P}})$ as a submodule of S^f. We use this observation in Section 3.

Definition 2.3 (*[14], Section 8.1*)**.** A *shelling* of a pure k-dimensional polytopal complex $\mathcal{P}$ is a linear ordering $P_1, P_2, \ldots P_s$ of the facets of $\mathcal{P}$ such that either $\mathcal{P}$ is zero dimensional or the following conditions are satisfied.

(1) The boundary complex δP_1 of the first facet P_1 has a shelling.
(2) For $1 < j \leq s$ the intersection of the facet P_j with the previous facets is nonempty and is a beginning segment of a shelling of the $(k-1)$-dimensional boundary complex of P_j.

A complex $\mathcal{P}$ is *shellable* if it has a shelling.

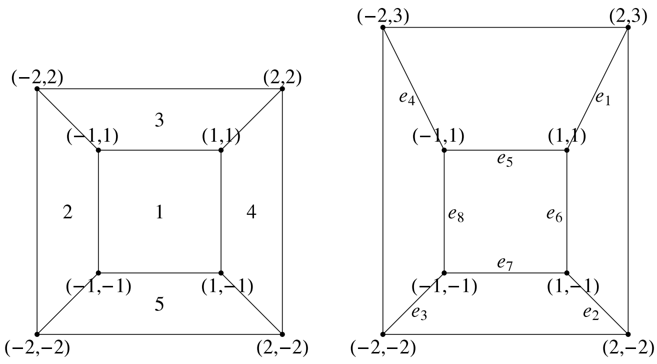
When Δ is simplicial, shellability of Δ implies the Stanley–Reisner ring A_Δ is Cohen–Macaulay ([8] Theorem 13.45). Since the variables of S form a system of parameters for $C^0(\hat{\Delta})$, Theorem 13.37 of [8] implies that $C^0(\hat{\Delta})$ is free as an S-module iff $C^0(\hat{\Delta})$ is Cohen–Macaulay. Billera's result in [4] that $A_\Delta \cong C^0(\hat{\Delta})$ then yields that shellability of Δ implies freeness of $C^0(\hat{\Delta})$ for simplicial Δ. In the next section we see that this statement fails for polytopal complexes.

3. The counterexample

When Δ is simplicial the isomorphism $A_\Delta \cong C^0(\hat{\Delta})$ implies that $C^0(\hat{\Delta})$ is a combinatorial object. This is not true for polytopal complexes: Example 1.1 of [10] displays a pair of pure, two dimensional, combinatorially equivalent polytopal complexes $P_1, P_2 \subset \mathbb{R}^2$ such that the Hilbert polynomials $HP(C^0(\hat{P_1}), k)$ and $HP(C^0(\hat{P_2}), k)$ are not equal. These complexes are shellable and $C^0(\hat{P_1})$ and $C^0(\hat{P_2})$ are both free.

In [13], Yuzvinsky exhibits combinatorially equivalent polytopal complexes $P, Q \subset \mathbb{R}^2$, both pure of dimension two and homotopic to a circle, so that $C^0(\hat{P})$ is free while $C^0(\hat{Q})$ is not. However, pure d-complexes with nontrivial singular homology in dimension $<d$ are not shellable. Example 3.1 below shows that even for shellable polytopal complexes $\mathcal{P} \subset \mathbb{R}^2$, freeness of $C^0(\hat{\mathcal{P}})$ may depend on the embedding.

Example 3.1. $\mathcal{Q}_1, \mathcal{Q}_2 \subset \mathbb{R}^2$ below are two combinatorially equivalent embeddings of a two dimensional complex $\mathcal{Q}$ with $f = 5$, $e^0 = 8$, and $v^0 = 4$. The numbering on the facets of $\mathcal{Q}_1$ gives a shelling order.



(a) $\mathcal{Q}_1$. (b) $\mathcal{Q}_2$.

Using the description $C^0(\hat{\mathcal{P}}) = \ker \phi$ of Proposition 2.2 and projecting onto $S^f = S^5$ we find explicit generators for $C^0(\hat{\mathcal{Q}}_1)$ and $C^0(\hat{\mathcal{Q}}_2)$ as submodules of S^5. A free basis for $C^0(\hat{\mathcal{Q}}_1)$ is given by the columns of the following matrix. Each row corresponds to a facet of $\mathcal{Q}_1$ listed in the same order as the shelling above.

$$\begin{pmatrix} 1 & y+z & (y+z)(x-z) & 0 & 0 \\ 1 & y-x & 2z(x-y) & (x+z)(x-y) & 0 \\ 1 & 2y & 2z(x-y) & (x-y)(z-y) & (x+y)(x-y)(y-z) \\ 1 & x+y & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now let $l_1 = y-2x+z, l_2 = x+y, l_3 = y-x, l_4 = y+2x+z, l_5 = y-z, l_6 = x-z, l_7 = y+z$, and $l_8 = x+z$ (l_i corresponds to the edge e_i in $\mathcal{Q}_2$ above). A minimal set of generators for $C^0(\hat{\mathcal{Q}}_2)$ is given by the columns of the matrix below.

$$\begin{pmatrix} 1 & 2l_7 & -4l_6l_7 & 0 & 2l_6l_7^2 & 2l_6l_7l_8 \\ 1 & 2l_3 & -8xl_3 & 0 & -4zl_3l_7 & 0 \\ 1 & 3y+z & (3y+z)l_1 & -l_1l_4l_5 & -z(3y+z)l_1 & -zl_1l_4 \\ 1 & 2l_2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If an S-module M is free on 6 generators it is readily seen that the leading coefficient of the Hilbert polynomial $HP(M, k)$ is 3. However it follows from [2] Theorem 4.5 that the leading coefficient of $HP(C^0(\hat{\mathcal{Q}}_2), k)$ is $f/2 = 5/2$. Hence $C^0(\hat{\mathcal{Q}}_2)$ is not free. Macaulay 2 code to perform the computations above is available at http://math.illinois.edu/~dipasqu1.

Let $\mathcal{P}^n$ denote the nth iterate of the coning construction over $\mathcal{P}$, starting with $\mathcal{P}^1 = \hat{\mathcal{P}}$. The complexes $\mathcal{Q}_1^n, \mathcal{Q}_2^n \subset \mathbb{R}^{n+2}$ for $n \geq 1$ are combinatorially equivalent and shellable since coning preserves shellability. If $C^0(\mathcal{P})$ is graded, Theorem 6.3 of [4] shows that freeness of $C^0(\hat{\mathcal{P}})$ is equivalent to freeness of $C^0(\mathcal{P})$. Since both $C^0(\hat{\mathcal{Q}}_1)$ and $C^0(\hat{\mathcal{Q}}_2)$ are graded, it follows that $C^0(\mathcal{Q}_1^n)$ is free and $C^0(\mathcal{Q}_2^n)$ is not free for $n \geq 1$. This establishes Theorem 1.2.

4. Nonfreeness and Ext

We now use results from [12] to indicate more precisely where the nonfreeness of $C^0(\hat{\mathcal{Q}}_2)$ arises. Let all notations be the same as in Section 2. Note that by the exact sequence of Proposition 2.2, $C^0(\hat{\mathcal{P}})$ is a second syzygy module over $S = \mathbb{R}[x, y, z]$ hence can have projective dimension at most 1 by the Hilbert Syzygy Theorem. So $C^0(\hat{\mathcal{P}})$ is free iff $Ext_S^1(C^0(\hat{\mathcal{P}}), S) = 0$.

Our strategy is to express $Ext_S^1(C^0(\hat{\mathcal{P}}), S)$ as an Ext^3 of a simpler module with an explicit presentation given in [12]. This analysis is then applied to the complex $\mathcal{Q}_2$ from Example 3.1 to show that $Ext_S^1(C^0(\hat{\mathcal{Q}}_2), S) \neq 0$ and hence $C^0(\hat{\mathcal{Q}}_2)$ is not free. To do this we recall a fundamental construction in spline theory.

For a k-dimensional face ξ of $\hat{\mathcal{P}}$, let $I_\xi \subset S$ denote the ideal of the linear span of ξ in $\mathbb{R}^3$. Also let $\mathcal{P}_i^0$ denote the internal i-faces of $\mathcal{P}$. In [3], Billera defines the following complex. For higher orders of smoothness, a variant is introduced in [12], but we will not need this.

Definition 4.1. For a polytopal 2-complex $\mathcal{P} \subset \mathbb{R}^2$, let $\mathcal{C} = C.$ be the complex of $S = \mathbb{R}[x, y, z]$ modules with cellular differential $\delta_i : C_i \to C_{i-1}$

$$0 \to \bigoplus_{\sigma \in \mathcal{P}_2} S \xrightarrow{\delta_2} \bigoplus_{\tau \in \mathcal{P}_1^0} S/I_\tau \xrightarrow{\delta_1} \bigoplus_{v \in \mathcal{P}_0^0} S/I_v \to 0.$$

Let $H_*(\mathcal{C})$ be the homology of $\mathcal{C}$. By construction, $C^0(\hat{\mathcal{P}}) = H_2(\mathcal{C})$. The following proposition is immediate.

Proposition 4.2. *The complex $\mathcal{C}$ naturally fits into a short exact sequence of complexes $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ as below:*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
\mathcal{A} & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{\tau\in\mathcal{P}_1^0} I_\tau \arrow[r, "\delta_1''"] \arrow[d] & \bigoplus_{v\in\mathcal{P}_0^0} I_v \arrow[r] \arrow[d] & 0 \\
\mathcal{B} & 0 \arrow[r] & \bigoplus_{\sigma\in\mathcal{P}_2} S \arrow[r, "\delta_2'"] \arrow[d] & \bigoplus_{\tau\in\mathcal{P}_1^0} S \arrow[r, "\delta_1'"] \arrow[d] & \bigoplus_{v\in\mathcal{P}_0^0} S \arrow[r] \arrow[d] & 0 \\
\mathcal{C} & 0 \arrow[r] & \bigoplus_{\sigma\in\mathcal{P}_2} S \arrow[r, "\delta_2"] \arrow[d] & \bigoplus_{\tau\in\mathcal{P}_1^0} S/I_\tau \arrow[r, "\delta_1"] \arrow[d] & \bigoplus_{v\in\mathcal{P}_0^0} S/I_v \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

Observe that $H_*(\mathcal{B}) = H_*(\mathcal{P}, \partial\mathcal{P}; S)$, the cellular homology of $\mathcal{P}$ relative to its boundary $\partial\mathcal{P}$ with coefficients in S. In particular, if $\mathcal{P}$ is topologically a disk then $H_*(\mathcal{P}, \partial\mathcal{P}; S) \cong \widetilde{H}_*(\mathbb{S}^2; S)$ ($\mathbb{S}^2$ being the 2-sphere) via excision and the universal coefficient theorem. Hence $H_2(\mathcal{B}) = S$ is the only nonzero homology module of $\mathcal{B}$.

Proposition 4.3. *Let the complexes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be as above and suppose that $\mathcal{P} \subset \mathbb{R}^2$ is a topological disk. Then $Ext_S^1(C^0(\hat{\mathcal{P}}), S) \cong Ext_S^3(H_0(\mathcal{A}), S)$.*

Proof. We first establish that $Ext_S^1(C_0(\hat{\mathcal{P}}), S) \cong Ext_S^3(H_1(\mathcal{C}), S)$. Recall $M = \operatorname{coker}\phi$ where ϕ is the matrix in Proposition 2.2. Since $C^0(\hat{\mathcal{P}})$ is a second syzygy module for M, $Ext_S^3(M, S) = Ext_S^1(C^0(\hat{\mathcal{P}}), S)$. We have the following short exact sequence ([12], Lemma 3.1) relating M and $H_1(\mathcal{C})$:

$$0 \to H_1(\mathcal{C}) \to M \to \bigoplus_{v\in\mathcal{P}_0} S/I_v \to 0.$$

From the long exact sequence in Ext and the fact that $Ext_S^3(S/I_v, S) = 0$ we obtain $Ext_S^3(M, S) \cong Ext_S^3(H_1(\mathcal{C}), S)$. Now we isolate a piece of the long exact sequence coming from the short exact sequence of Proposition 4.2:

$$\cdots \to H_1(\mathcal{B}) \to H_1(\mathcal{C}) \to H_0(\mathcal{A}) \to H_0(\mathcal{B}) \to \cdots$$

$H_*(\mathcal{B}) \cong \widetilde{H}_*(\mathbb{S}^2; S)$, so $H_1(\mathcal{B}) = H_0(\mathcal{B}) = 0$, yielding $H_1(\mathcal{C}) \cong H_0(\mathcal{A})$. $\square$

We now show a useful presentation of $H_0(\mathcal{A})$. Call an edge τ *totally interior* if both its vertices are interior and for $v \in \mathcal{P}_0^0$ let $Syz(v)$ be the graded submodule of $\bigoplus_{\tau\in\mathcal{P}_1^0} Se_\tau$ consisting of syzygies of forms l_τ for those edges τ containing v, i.e. $Syz(v) = \left\{\sum_{v\in\tau} a_\tau e_\tau \,|\, \sum_{v\in\tau} a_\tau l_\tau = 0, \text{ for } a_\tau \in S\right\}$. In [12] Schenck and Stillman give the following presentation for $H_0(\mathcal{A})$.

Lemma 4.4. *Define $K \subset \bigoplus_{\tau\in\mathcal{P}_1^0} Se_\tau$ to be the submodule generated by*

$$\{e_\tau | \tau \text{ not totally interior}\}$$

and $\bigcup_{v\in\mathcal{P}_0^0} Syz(v)$. The S-module $H_0(\mathcal{A})$ is given by generators and relations by

$$0 \to K \to \bigoplus_{\tau\in\mathcal{P}_1^0} Se_\tau \to H_0(\mathcal{A}) \to 0.$$

We make this presentation more explicit. If d edges are incident at a vertex v, $Syz(v)$ is generated by $d-2$ relations of degree zero and a single relation of degree one. Let α denote the number of $\tau \in \mathcal{P}_1^0$ which are not totally interior.

Corollary 4.5. *Order the edges of $\mathcal{P}$ so that those which are not totally interior occur last. The S-module $H_0(\mathcal{A})$ has presentation*

$$S^h \xrightarrow{N} S^{e^0} \to H_0(\mathcal{A}) \to 0,$$

where h is the number of columns of N and N has block decomposition

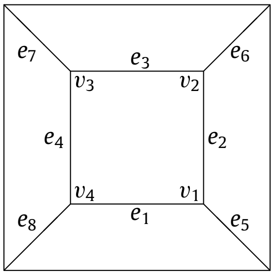
$$N = \left(\begin{array}{c|c|c} B_0 & B_1 & M \end{array} \right).$$

The columns of B_0 run over syzygies of degree zero and the columns of B_1 run over syzygies of degree one at each $v \in \mathcal{P}_0^0$, and $M = \begin{pmatrix} 0 \\ id_\alpha \end{pmatrix}$, where id_α is the $\alpha \times \alpha$ identity matrix. Pruning out id_α gives the more compact presentation

$$S^{h-\alpha} \xrightarrow{N'} S^{e^0-\alpha} \to H_0(\mathcal{A}) \to 0$$

where $N' = \left(B_0' | B_1'\right)$ is obtained from N by deleting M and the rows corresponding to e_τ with τ not totally interior.

We illustrate this corollary by applying it to the complex $\mathcal{Q}$ from Section 3 with a generic embedding into $\mathbb{R}^2$.



Let the linear form l_i correspond to e_i. For each interior vertex v, $Syz(v)$ is generated by a degree zero syzygy and a degree one syzygy. Let the degree zero syzygies at v_1, v_2, v_3, and v_4 respectively be $a_1e_1+b_1e_2+c_1e_5$, $a_2e_2+b_2e_3+c_2e_6$, $a_3e_3+b_3e_4+c_3e_7$, and $a_4e_4+b_4e_1+c_4e_8$ for some constants $a_i, b_i, c_i \in \mathbb{R}$. Similarly choose linear syzygies $l_5e_1 - l_1e_5$, $l_6e_2 - l_2e_6$, $l_7e_3 - l_3e_7$, and $l_8e_4 - l_4e_8$ at each interior vertex. Using $id_\alpha = id_4$ to eliminate the rows corresponding to the not totally interior edges e_5, e_6, e_7, e_8 gives the presentation below, where the ith row of N' corresponds to e_i and the ith columns of B'_0 and B'_1 correspond to v_i.

$$S^8 \xrightarrow{N'} S^4 \rightarrow H_0(\mathcal{A}) \rightarrow 0$$

$$N' = (B'_0|B'_1) = \left(\begin{array}{cccc|cccc} a_1 & 0 & 0 & b_4 & l_5 & 0 & 0 & 0 \\ b_1 & a_2 & 0 & 0 & 0 & l_6 & 0 & 0 \\ 0 & b_2 & a_3 & 0 & 0 & 0 & l_7 & 0 \\ 0 & 0 & b_3 & a_4 & 0 & 0 & 0 & l_8 \end{array} \right).$$

Clearly $\operatorname{rank}(B'_0) \geq 3$ since a_i, b_i, c_i are all nonzero (otherwise l_i would not define distinct lines). If $\operatorname{rank}(B'_0) = 4$ then $H_0(\mathcal{A}) = 0$. If $\operatorname{rank}(B'_0) = 3$ then the presentation above can be simplified to $S^4 \xrightarrow{N''} S \rightarrow H_0(\mathcal{A}) \rightarrow 0$, where $N'' = \begin{pmatrix} l_5 & l_6 & l_7 & l_8 \end{pmatrix}$. So in the rank 3 case $H_0(\mathcal{A}) \cong S/(l_5, l_6, l_7, l_8)$. It is easy to check that both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ from Example 3.1 fall into the rank 3 case.

Let the complex $\mathcal{A}_i$ (as in Proposition 4.2) correspond to the embedding $\mathcal{Q}_i$ for $i = 1, 2$. In the case of $\mathcal{Q}_1$, $l_5 = l_7 = x + y$ and $l_6 = l_8 = x - y$. Hence $H_0(\mathcal{A}_1) \cong S/(x, y)$. But $Ext^3_S(S/(x, y), S) = 0$ so $C^0(\hat{\mathcal{Q}}_1)$ is free by Proposition 4.3 since $Ext^i_S(C^0(\hat{\mathcal{Q}}_1), S) = 0$ for $i > 1$.

In the case of $\mathcal{Q}_2$, $l_5 = y + x$, $l_6 = y - 2x + z$, $l_7 = y + 2x - z$, and $l_8 = y - x$. So $H_0(\mathcal{A}_2) \cong S/(x, y, z)$ and $Ext^3_S(S/(x, y, z), S) = \mathbb{R} \neq 0$. Proposition 4.3 then implies that $C^0(\hat{\mathcal{Q}}_2)$ is not free. These computations confirm the claims in Example 3.1 and give a more suggestive reason for the nonfreeness of $C^0(\hat{\mathcal{Q}}_2)$.

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Further reading

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